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On Fermat's principle and Snell's law in lossy anisotropic media

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ABSTRACT

Fermat's principle of least action is one of the methods used to trace rays in inhomogeneous media. Its form is the same in anisotropic elastic and anelastic media, with the difference that the velocity depends on frequency in the latter case. Moreover, the ray, envelope, and energy velocities replace the group velocity because this concept has no physical meaning in anelastic media. We have first considered a lossy (anelastic) anisotropic medium and established the equivalence between Fermat's principle and Snell's law in homogeneous media. Then, we found that the different ray velocities defined in the literature were the same for stationary rays in homogeneous media, with phase and inhomogeneity angles satisfying the principle and the law. We considered an example of a transversely isotropic medium with a vertical symmetry axis and wavelike and diffusionlike properties. In the first case, the differences were negligible, which was the case of real rocks having a quality factor greater than five. Strictly, ray tracing should be based on the so-called stationary complex slowness vector to obtain correct results, although the use of homogeneous viscoelastic waves (zero inhomogeneity angle) is acceptable as an approximation for earth materials. However, from a rigorous point of view, the three velocities introduced in the literature to define the rays present discrepancies in heterogeneous media, although the differences are too small to be measured in earth materials. The findings are also valid for electromagnetic waves by virtue of the acoustic-electromagnetic analogy.

INTRODUCTION

Ray-tracing methods are used in several applied fields, such as seismology (Rawlinson et al., 2007), quantum mechanics (Synge, 1954), and electromagnetism (Glassner, 1989). An exhaustive review in seismology (anelastic media) is given by Thomson (1997) and Hanyga and Seredyńska (2000). The algorithm used for ray bending at interfaces can be based on Fermat's principle (e.g., Moser, 1991; Červený, 2002) or Snell's law (Hanyga and Seredyńska, 2000), where the first approach considers the calculation of the shortest path with appropriate ray (group or energy) velocities (Fermat's principle), whereas the second approach is based on the continuity of the projection of the (complex) slowness components on the interfaces (Snell's law). It is well known that Fermat's principle and Snell's law are equivalent in isotropic media. The equivalence in lossless and lossy anisotropic media using ray and phase velocities and angles has not been fully clarified. We prove the equivalence in this work. On the other hand, the problem becomes more complex if the ray tracing considers real rays involving Fermat's principle based on ray velocities. Many authors use the group velocity — a kinematic concept — although this quantity has no physical meaning in strongly lossy media; the more general physical envelope and energy velocities should be used, the latter being a dynamical concept based on the Umov-Poynting flux vector (Carcione et al., 1996; Carcione, 2015).

A technique used to trace rays is the method of stationary phase, introduced by Kelvin and Thomson (1887), and it is based on the group velocity approximation (Havelock, 1914). Waves in anelastic media are in general inhomogeneous; i.e., the propagation and attenuation directions do not coincide. Hearn and Krebes (1990) use the method of steepest descent to approximate the integral giving the wavefield at the observation point. In this way, they deduce the value of the initial propagation and attenuation angles from the value of the complex ray parameter at the saddle point of the

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complex traveltime function (see also Hanyga and Seredyńska, 2000). The ray determined by the saddle point is termed the "stationary ray" and has the smallest traveltime based on the complex phase, a result which is consistent with Fermat's principle. This traveltime is based on kinematic — phase — considerations rather than energy-flux quantities.

Recently, Vavryčuk (2006, 2007, 2008, 2010) improves the theory of ray tracing by introducing energy-based quantities. The equations, which hold for smoothly inhomogeneous anisotropic low-loss viscoelastic media, are based on real-valued rays defined as trajectories based on an inhomogeneous complex and stationary slowness vector, where the complex energy velocity is homogeneous in uniform media. It is shown here that his energy velocity is equivalent to Hearn and Krebes's ray velocity and to the energy velocity defined by Carcione (2015) in homogeneous media if stationary slowness is used.

In this work, we investigate the equivalence between Fermat's principle and Snell's law in terms of the ray and phase velocities and angles. We show the equivalence for waves in lossy media explicitly. In the lossy case, the ray velocity compatible with Fermat's principle and Snell's law is that defined with the stationary complex slowness. In this case, the kinematic (envelope velocity) and energy definitions of velocity along the raypath are equivalent.

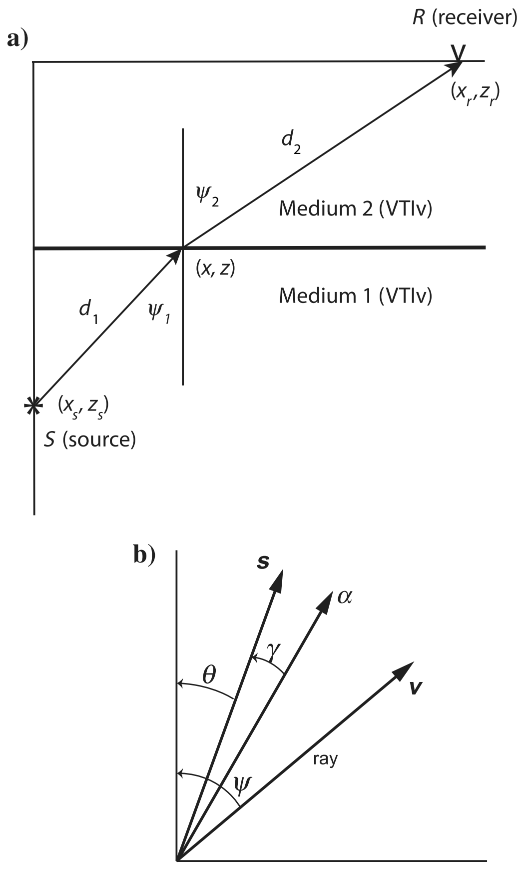

Figure 1. (a) Refraction experiment and (b) characteristics of an inhomogeneous viscoelastic wave. The source and receiver positions are known, as well as the location of the interface (z). The media are transversely isotropic (TI) and viscoelastic (v) with a vertical symmetry axis (V). The slowness, attenuation, and energy-velocity vectors (**s**, **α**, and **v**) point out in different directions.

We consider examples in a transversely isotropic medium with a vertical symmetry axis and transmission through a single material interface, and we show that Fermat's principle and Snell's law are equivalent for stationary slowness, at least in homogeneous media. For the attenuation values found in earth rocks, the differences between the kinematic and dynamical approaches, based on a nonstationary complex slowness, are small. Here, we consider extreme values that allow us to see the differences.

FERMAT'S PRINCIPLE AND SNELL'S LAW

Let us first define the rheology, i.e., the stiffness coefficients relating stress and strain. The following equations are taken from Carcione (2015). The incidence and transmission media are defined by the stiffnesses p_{IJ} and p'_{IJ} and densities ρ and ρ', where

$$p_{IJ} = c_{IJ} M_{IJ}(\omega), \qquad I, J = 1, \ldots, 6, \tag{1}$$

where c_{IJ} are the unrelaxed elasticity constants (real quantities) in the lossless case, M_{IJ} are the anelasticity kernels (complex and frequency-dependent), and ω is the angular frequency. A similar expression holds for p'_{IJ}.

The simplest realistic model, but general enough for our purposes, is a single Zener element, which represents a typical relaxation peak. It can be expressed as

$$M_{IJ}(\omega) = \frac{\sqrt{1+Q_{IJ}^2} + \mathrm{i}\omega Q_{IJ}\tau_0 - 1}{\sqrt{1+Q_{IJ}^2} + \mathrm{i}\omega Q_{IJ}\tau_0 + 1} = \frac{\mathrm{i}\omega\tau_0 + 1/a}{\mathrm{i}\omega\tau_0 + a}, \tag{2}$$

where Q_{IJ} is the lowest value of the quality factor (a measure of wave loss) at the frequency $\omega_0 = 1/\tau_0$. The high-frequency limit corresponds to the elastic case, with $M_{IJ} \to 1$. It can be shown that the quantity a is the ratio between the unrelaxed velocity and the relaxed velocity.

Let us consider the source-receiver configuration shown in Figure 1, where **v** can be the ray, envelope, or energy velocities (ray velocities in general), according to Vavryčuk (2007, 2010), Hearn and Krebes (1990), and Carcione (2015), respectively. Postma (1955) gives a demonstration of the expression of the envelope velocity. It is shown here that this velocity is that implicit in the theory of Hearn and Krebes (1990). The problem consists in finding the velocity that gives the minimum traveltime between S and R. This involves finding the point x. The traveltime (a real quantity) is given by

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2}, \tag{3}$$

where

$$d_1 = \sqrt{(x-x_s)^2 + (z-z_s)^2}, \qquad d_2 = \sqrt{(x_r-x)^2 + (z_r-z)^2}, \tag{4}$$

and v_1 and v_2 are the ray velocities, both equal to the group velocities in the lossless case (Carcione, 2015). Note that these velocities are frequency dependent and the analysis is therefore performed for a given frequency. In the elastic case, there is one velocity because all the Fourier components travel with the same velocity.

The condition of minimum time $dt/dx = 0$ yields

$$\frac{x - x_s}{v_1 d_1} - \frac{d_1}{v_1^2}\frac{dv_1}{dx} + \frac{x_r - x}{v_2 d_2} - \frac{d_2}{v_2^2}\frac{dv_2}{dx} = 0. \tag{5}$$

It is $dv_1/dx = (dv_1/d\psi_1)(d\psi_1/dx)$, $\tan\psi_1 = (x - x_s)/(z - z_s)$, $d\psi_1/dx = \cos^2\psi_1/(z - z_s)$, and $\cos\psi_1 = (z - z_s)/d_1$, implying $d\psi_1/dx = \cos\psi_1/d_1$. Similarly, it can be shown that $d\psi_2/dx = -\cos\psi_2/d_2$. Then, we have

$$\frac{\sin\psi_1}{v_1} - \frac{dv_1}{d\psi_1}\frac{\cos\psi_1}{v_1^2} = \frac{\sin\psi_2}{v_2} - \frac{dv_2}{d\psi_2}\frac{\cos\psi_2}{v_2^2} \equiv F(\omega), \tag{6}$$

which is Fermat's principle. Point x is found by minimizing the traveltime (equation 3), considering that $v_1 = v_1(\psi_1)$ and $v_2 = v_2(\psi_2)$. Because the velocities are frequency dependent, point x differs for each frequency.

On the other hand, it is well known that Snell's law is

$$\frac{\sin\theta_1}{v_{p1}} = \frac{\sin\theta_2}{v_{p2}} \equiv S(\omega), \tag{7}$$

where v_p and θ are the phase velocity and angle, respectively (e.g., Carcione, 2015).

The ray velocity has different interpretations in the literature. Appendix A introduces the less known concept of envelope velocity $v_{\rm env}$ in the case of lossy anisotropic media (see Carcione [2015], sections 1.4.3 and 4.6.3), whereas the energy velocity used in this work is that of Carcione (2015) and is defined as the average Umov-Poynting vector divided by the total average energy (see Auld, 1990). In Appendix B, we show the equivalence between Fermat's principle and Snell's law in general ($F = S$). To our knowledge, no demonstration has been given of this equality, although it is frequently stated that Snell's law is equivalent to Fermat's principle. Appendices C and D summarize the expressions for the phase (v_p), group (v_g) envelope, and energy (v_e) velocities of SH and qP-qS waves introduced by Carcione and Cavallini (1993) (see Carcione, 2015) for homogeneous and inhomogeneous viscoelastic waves.

Hearn and Krebes (1990) show that Snell's law and Fermat's principle are satisfied for the stationary ray (isotropic media). It is shown in Appendix E that the ray velocity involved in their calculations ($v_{\rm HK}$) is precisely the envelope velocity for any arbitrary value of the inhomogeneity angle. On the other hand, Appendix F summarizes the approach of Vavryčuk (2007, 2010), who defines the ray velocity $v_{\rm ray}$ corresponding to a stationary slowness vector. It is shown that his ray velocity is equivalent to the envelope velocity, and the concepts of stationary ray and stationary slowness vector are the same (homogeneous media). Vavryčuk (2007, 2010) shows that if the stationary slowness vector is replaced into the energy-velocity vector $\mathbf{v}_e$ (equation C-4), its magnitude is $v_{\rm ray}$. A demonstration is reported in Appendix F for the SH-wave. Then,

$$v_{\rm HK} = v_{\rm env} = v_{\rm ray} = v_e \tag{8}$$

for the stationary slowness and homogeneous media. For arbitrary values of θ and γ (nonstationary slowness), it is $v_{\rm HK} = v_{\rm env} \neq v_{\rm ray} \neq v_e$, even in homogeneous media.

EXAMPLE

Let us consider SH-waves and a homogeneous viscoelastic transversely isotropic medium with the following properties: $c_{55} = 1$ GPa, $c_{66} = 2$ GPa, $Q_{55} = 1$, $Q_{66} = 1000$, and $\rho = 2.1$ g/cm^3. The effects of anelasticity are significant when attenuation is strong, which occurs for diffusion Q's, say, $Q < 5$. For wavelike values of Q, there are no differences in practice between all the ray velocities for a nonstationary slowness vector, i.e., for any arbitrary value of the phase and inhomogeneity angles. Note that real rocks have a Q value as low as five, and this occurs for shallow marine sediments (e.g., Hamilton, 1972).

We take $\omega\tau_0 = 1$ in equation 2, i.e., the peak frequency. In this case, equation 1 becomes

$$p_{IJ} = \bar{c}_{IJ}(1 + {\rm i}Q_{IJ}^{-1}),$$
$$\bar{c}_{IJ} = c_{IJ}\left(1 + Q_{IJ}^{-2} + Q_{IJ}^{-1}\sqrt{1 + Q_{IJ}^{-2}}\right)^{-1}, \tag{9}$$

where it is clear that when $Q_{IJ} \to \infty$, $p_{IJ} \to c_{IJ}$, and $\bar{c}_{IJ} = c_{IJ}$. For a finite-quality factor, we have $\bar{c}_{IJ} < c_{IJ}$.

It is shown in Appendices E and F that Hearn and Krebes (1990) and Vavryčuk (2007) concepts of the stationary ray are equivalent and that the real ray velocity involved in their methods is the same as the envelope velocity defined here and is equal to the energy velocity introduced by Carcione (2015) if stationary slowness is used, i.e., the horizontal slowness component, which minimizes the traveltime from source to interface (the homogeneous medium). This stationary slowness for SH-waves is given by equation F-6. Figure 2 shows the ray velocity (Figure 2a) and angles (Figure 2b) as a function of the horizontal distance $(x - x_s)$ (from source to interface), corresponding to the stationary slowness. In this case, all of the velocities are the same. On the other hand, Figure 3 shows the same results for a nonstationary slowness ($\gamma = 0$). It is clear that the velocities differ, although for real rocks (wavelike Q values, $Q_{55} > 5$), it can be shown that the differences are negligible. For increasing γ, the velocity $v_{\rm ray}$ shows big differences at near distances with respect to the stationary velocity.

In the case of the interface problem shown in Figure 1, the phase and inhomogeneity angles of the upper medium are defined by Snell's law. Let us consider, for example, the refraction of an inhomogeneous SH-wave from medium 1 to medium 2, where, from Snell's law, s_x given by equation C-4 is the horizontal complex slowness of both media. Because the dispersion relation in medium 2 is (Carcione, 2015)

$$p'_{55}s'^2_z + p'_{66}s_x^2 - \rho' = 0, \tag{10}$$

we have

$$s'_z = \sqrt{\frac{\rho' - p'_{66}s_x^2}{p'_{55}}}. \tag{11}$$

From equations C-4, we obtain

$$\tan\theta_2 = \frac{s_{xR}}{s'_{zR}} \quad \text{and} \quad \tan(\theta_2 + \gamma_2) = \frac{s_{xI}}{s'_{zI}}. \tag{12}$$

Having the angles, a, b, c, and q can be obtained and with these the whole set of properties in equation C-4. Also, the phase velocity and attenuation factor can be determined as

$$v'_p = \frac{\sin\theta_2}{s_{xR}}, \qquad \bar{\alpha}' = -\frac{s_{xI}}{\sin(\theta_2 + \gamma_2)}. \tag{13}$$

The same equations 10–13 can be used for the incidence medium, using its corresponding properties.

Let us consider an specific example, where $(x_s, z_s) = (0{,}70)$ m, $(x_r, z_r) = (80{,}0)$ m, $z = 40$ m, and the properties in Table 1. We solve equation E-4 by stepwise iteration, using the downhill method (Bach, 1969). The function to be solved must be analytic in the region, where the root is being sought. The solution gives the stationary (complex) slowness component $s_x = (0.6038, -0.05278)$ s/km and the values shown in Table 2, where the ray angles have been obtained as

$$\tan\psi_{e1} = \frac{\mathrm{Re}(p_{66}s_x)}{\mathrm{Re}(p_{55}s_z)}, \qquad \tan\psi_{\mathrm{ray1}} = \mathrm{Re}\left(\frac{p_{66}s_x}{p_{55}s_z}\right), \tag{14}$$

and

$$\tan\psi_{e2} = \frac{\mathrm{Re}(p'_{66}s_x)}{\mathrm{Re}(p'_{55}s'_z)}, \qquad \tan\psi_{\mathrm{ray2}} = \mathrm{Re}\left(\frac{p'_{66}s_x}{p'_{55}s'_z}\right). \tag{15}$$

In the case of a source and receiver located in the same (homogeneous) medium, we have $\psi_e = \psi_{\mathrm{ray}}$ because the ray velocity (equation F-1) is an homogeneous vector (see equation F-6). In this case, the energy-velocity vector (equation C-4) and ray velocity (equation F-3) are the same. However, in the case that the source and receiver are located in different media, the ray and energy velocity vector are not homogeneous and do not have the same value, as can be seen in the previous calculations. According to Figure 1, the point where the ray crosses the interface, can be obtained as

$$x = x_s + (z_s - z)\tan\psi_{\mathrm{ray1}} \quad \text{or}$$
$$\bar{x} = x_r - (z - z_r)\tan\psi_{\mathrm{ray2}}, \tag{16}$$

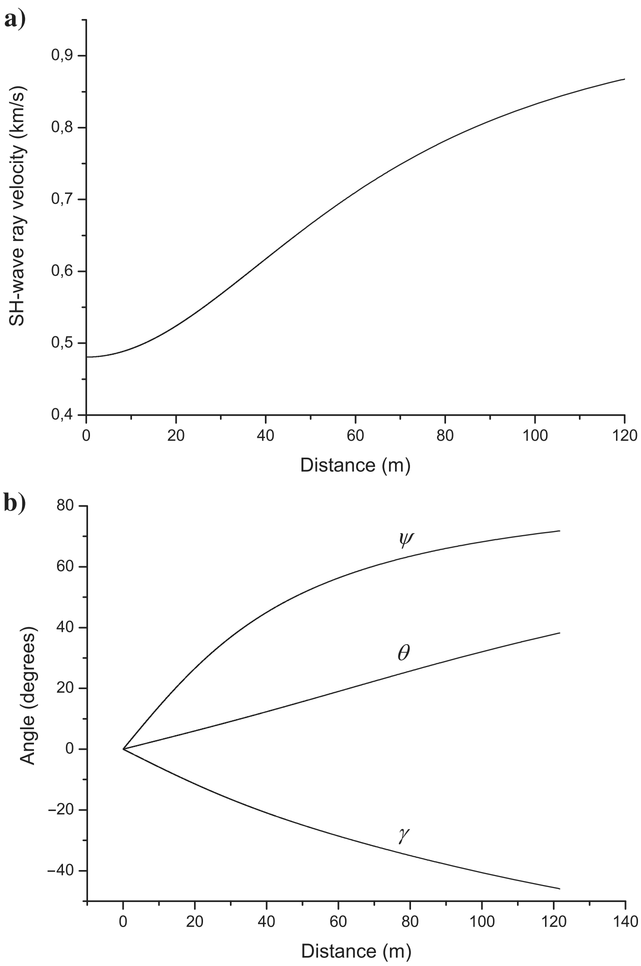

Figure 2. (a) SH-wave ray velocity and (b) angles (ray ψ, phase θ, and inhomogeneity γ angles) as a function of the horizontal distance $(x - x_s)$, corresponding to the stationary slowness. All of the ray velocities defined in this work are the same for the stationary ray; i.e., $v_{\mathrm{HK}} = v_{\mathrm{env}} = v_{\mathrm{ray}} = v_e$.

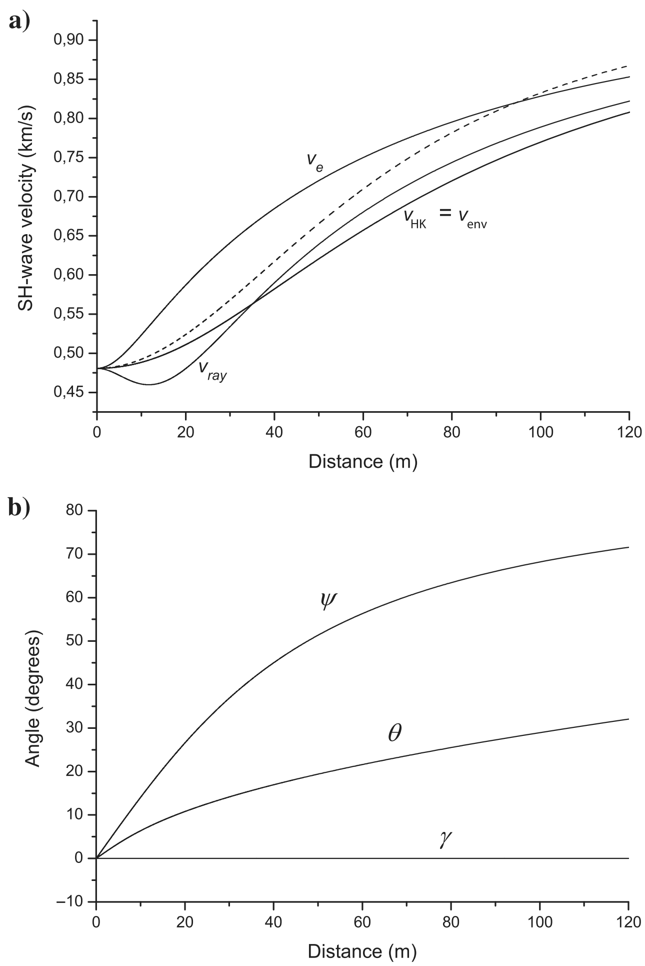

Figure 3. (a) SH-wave ray velocity and (b) angles (ray ψ, phase θ, and inhomogeneity γ angles) as a function of the horizontal distance $(x - x_s)$, corresponding to a nonstationary slowness, with $\gamma = 0$. It is $v_{\mathrm{HK}} = v_{\mathrm{env}} \neq v_{\mathrm{ray}} \neq v_e$. The dashed line corresponds to the ray velocity of the stationary ray shown in Figure 2a.

which gives $x = \bar{x} = 27.02$ m. The minimum time is the real part of the expression (equation E-3) and gives 0.149 s. However, if we compute ψ_{ray} for both media using the form (equation A-8), i.e., $v_p = v_{\mathrm{ray}} \cos(\psi_{\mathrm{ray}} - \theta)$, we do not obtain those of equations 14 and 15. This means that v_{ray} does not satisfy equation A-8.

Complete results are shown in Tables 3 and 4, where the envelope velocity, computed with equation A-5, is also reported. We have considered the material properties in Table 1 and very dissimilar values for the loss parameters of medium 1 ($Q_{55} = 0.0001$; $Q_{66} = 1000$). The traveltimes corresponding to each velocity are obtained as

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2}, \quad d_1 = \frac{z_s - z}{\cos \psi_1}, \quad d_2 = \frac{z - z_r}{\cos \psi_2}. \tag{17}$$

Even for extreme values, the numbers are similar. The intersection point at which the ray crosses the interface is not unique in the case of v_{env} and v_e. It is unique for v_{ray}, but in this case, the traveltime does not coincide with the stationary traveltime as can be seen in the last two columns.

Strictly, the minimum traveltime corresponds to the arrival time of the wavefront, which in anelastic media is determined by the unrelaxed stiffness coefficients c_{IJ} (Carcione, 2015). That is, the high-frequency limit, where the behavior is purely elastic (lossless) and the phase (and group) velocities have their maximum value as a function of frequency and of the propagation angle. In the lossless case, $\mathbf{s} \cdot \mathbf{x} = \tau$ and $\mathbf{s} \cdot \mathbf{v} = 1$ are equivalent if $\mathbf{v} = \mathbf{v}_{\mathrm{env}} = \mathbf{x}/\tau$, the envelope velocity, where all the quantities are real (Postma, 1955; Carcione [2015], section 1.4.3).

The arguments presented here hold also for electromagnetic waves by invoking the acoustic-electromagnetic analogy (e.g., Carcione and Cavallini, 1995b; Carcione et al., 2014).

CONCLUSION

We consider a lossy anisotropic medium and analyze the equivalence between Fermat's principle and Snell's law. Moreover, it is shown that for homogeneous media, Hearn and Krebes's and Vavryčuk's concepts of the stationary ray are equivalent and that the real ray velocity involved in their methods is the same as the envelope velocity defined here and is equal to the energy velocity introduced by Carcione if the stationary slowness is used, i.e., the horizontal slowness component, which minimizes the traveltime from source to receiver. However, the

Table 1. Material properties.

Medium	(c_{55}, Q_{55}) (GPa, –)	(c_{66}, Q_{66}) (GPa, –)	ρ (g/cm^3)
Incidence	(1, 1)	(2, 2)	2.1
Transmission	(1.5, 1.5)	(3, 3)	2

Table 2. Quantities corresponding to the stationary slowness.

Medium	θ (°)	γ (°)	v_p (m/s)	v_e (m/s)	v_{ray} (m/s)	ψ_e (°)	ψ_{ray} (°)
1	18.32	−14.72	520.54	567.40	570.33	41.77	42.00
2	27.47	−20.19	764.00	846.82	845.08	53.02	52.94

Table 3. Quantities for the values are given in Table 1. The v_{ray}, v_{env}, and v_e correspond to Vavryčuk's, Hearn and Krebes's, and Carcione's approaches.

Velocity	$v(1)$ (m/s)	$\psi(1)$ (°)	$v(2)$ (m/s)	$\psi(2)$ (°)	x (m)	$\bar{x}$ (m)	Time (s)	True time (s)
v_{ray}	570.34	42.01	845.10	52.94	27.02302	27.02302	0.14934	0.14947
v_{env}	568.95	42.34	845.36	52.82	27.33737	27.27269	0.14951	"
v_e	567.40	41.77	846.82	53.02	26.79548	26.86770	0.14943	"

Table 4. Quantities for very dissimilar values in the incidence medium.

Velocity	$v(1)$ (m/s)	$\psi(1)$ (°)	$v(2)$ (m/s)	$\psi(2)$ (°)	x (m)	$\bar{x}$ (m)	Time (s)	True time (s)
v_{ray}	19.37	69.29	662.00	0.92	79.36390	79.36390	4.4401802	4.4084448
v_{env}	19.37	69.13	662.01	1.21	78.69649	79.16202	4.4084404	"
v_e	19.65	69.45	661.94	0.63	80.03045	79.56571	4.4084493	"

energy-velocity vector is homogeneous only in homogeneous media; i.e., it is not homogeneous if source and receiver lie in different media. The problem is far from being solved because some discrepancies between the different approaches regard the refraction point and the value of the minimum traveltime in the heterogeneous case, which requires more accurate calculations because even for very low values of the loss parameters, the properties show similar (not identical) values. It remains to verify the equivalence of Snell's law and Fermat's principle in the heterogeneous case. Comparisons with full-wave simulations are useless because of the too-small differences and the fact that by using diffusion Q's, the evaluation of traveltimes is impossible.

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APPENDIX A

ENVELOPE VELOCITY

Let us assume the (x, z)-plane and a harmonic inhomogeneous plane wave in an anisotropic and viscoelastic medium

$$\exp[\mathrm{i}\omega(t - \mathbf{s}\cdot\mathbf{x})], \quad \text{(A-1)}$$

where $\mathbf{s} = s_x\hat{\mathbf{e}}_1 + s_z\hat{\mathbf{e}}_3$ is the slowness vector and $\mathbf{x} = (x, z) = (x_1, x_3)$ is the position vector. According to Figure 1b, the slowness components are

$$s_x = s_R l_1 - \mathrm{i}\bar{\alpha} m_1, \quad l_1 = \sin\theta, \quad m_1 = \sin(\theta+\gamma),$$
$$s_z = s_R l_3 - \mathrm{i}\bar{\alpha} m_3, \quad l_3 = \cos\theta, \quad m_3 = \cos(\theta+\gamma), \quad \text{(A-2)}$$

where θ is the propagation angle, γ is the inhomogeneity angle, s_R is the real wavenumber, $\bar{\alpha}$ is the attenuation factor normalized by the angular frequency, and the subindices R and I denote real and imaginary parts, respectively. Substituting equation A-2 into the plane-wave kernel (equation A-1) gives

$$\exp[\mathrm{i}\omega(t - s_R(l_1x + l_3z))]\exp[-\omega\bar{\alpha}(m_1x + m_3z))]. \quad \text{(A-3)}$$

The first exponential defines the velocity of propagation. A definition of the wave surface is given by the envelope of the plane (Love [1944], p. 299):

$$l_1x + l_3z = v_p, \qquad (l_ix_i = v_p), \quad \text{(A-4)}$$

where $v_p = 1/s_R$ is the phase velocity. This is because the velocity of the envelope of plane waves at unit propagation time, which we call v_{env}, has the components

$$(v_{\mathrm{env}})_i = x_i = \frac{\partial v_p}{\partial l_i} \quad \text{(A-5)}$$

and

$$v_{\mathrm{env}} = \sqrt{x^2 + z^2}. \quad \text{(A-6)}$$

In anisotropic elastic media, the envelope velocity is equal to the group and energy velocities (see Carcione [2015], section 1.4.3).

Differentiating equation A-4 with respect to θ, squaring it and adding the results to the square of equation A-4, we get

$$v_{\mathrm{env}} = \sqrt{v_p^2 + \left(\frac{dv_p}{d\theta}\right)^2}. \quad \text{(A-7)}$$

Postma (1955) obtains this equation for a transversely isotropic elastic medium.

As in the lossless case, the following property holds from equation A-4:

$$v_p = v_{\mathrm{env}}\cos(\psi - \theta), \quad \text{(A-8)}$$

where

$$\tan\psi = \frac{x}{z} = \frac{\partial v_p/\partial l_1}{\partial v_p/\partial l_3}. \quad \text{(A-9)}$$

Equation A-8 is also satisfied by the energy velocity and inhomogeneous waves, substituting the angle in (equation A-8) by the energy angle (Carcione [2015], equation 4.112).

Moreover, combining equations A-7 and A-8, we obtain

$$\tan(\psi - \theta) = \frac{1}{v_p}\frac{dv_p}{d\theta}, \quad \text{(A-10)}$$

where ψ is the same given in equation A-9.

Although the group velocity v_g is commonly called the envelope velocity in the literature, they are not the same in attenuating media. Rather, the envelope velocity is equal to the phase velocity v_p in isotropic anelastic media. If $\gamma = 0$, the envelope, phase, and energy velocities are the same, whereas the group velocity has no physical meaning (Carcione et al., 2010; Carcione, 2015). In anisotropic anelastic media, all the velocities differ, even for $\gamma = 0$.

The envelope velocity, as well as the phase velocity, is a kinematical quantity, not involving the definition of an energy balance equation, contrary to the energy velocity. The effect of γ on the velocities is illustrated in Carcione and Cavallini (1995a, 1997).

APPENDIX B

EQUIVALENCE BETWEEN FERMAT'S PRINCIPLE AND SNELL'S LAW

Let us consider a lossy anisotropic medium. Equations A-7, A-8, and A-10 hold. Taking the derivative of equation A-8 with respect to θ gives

$$-\sin(\psi - \theta)\left(\frac{d\psi}{d\theta} - 1\right) = \frac{v_p}{v_{\mathrm{env}}}\left(\frac{1}{v_p}\frac{dv_p}{d\theta} - \frac{1}{v_{\mathrm{env}}}\frac{dv_{\mathrm{env}}}{d\psi}\frac{d\psi}{d\theta}\right). \quad \text{(B-1)}$$

Combining equations A-10 and B-1 and after some calculations yields (see Appendix B of Ursin and Hokstad, 2003)

$$\frac{1}{v_{\text{env}}}\frac{dv_{\text{env}}}{d\psi} = \tan(\psi - \theta). \tag{B-2}$$

Substituting this equation into equation 6 gives

$$F = \frac{\sin\psi}{v_{\text{env}}} - \frac{\cos\psi}{v_{\text{env}}}\tan(\psi-\theta) = \frac{\sin\theta}{v_{\text{env}}\cos(\psi-\theta)} = \frac{\sin\theta}{v_p} = S, \tag{B-3}$$

where we have used equation A-8. The envelope velocity can be replaced by the group velocity v_g in the lossless case. This demonstration holds for the same medium.

APPENDIX C

SH-WAVE EQUATIONS

Let us consider the SH-wave and omit, for simplicity, the primes and subindices 1 and 2 corresponding to each medium. The complex, phase, group, energy, and envelope velocities are denoted by v_c, v_p, v_g, v_e, and v_{env}, respectively.

Lossless anisotropic medium

These equations are taken from Postma (1955) and Carcione (2015):

$$\begin{aligned}
v_p &= \sqrt{\rho^{-1}(c_{66}l_1^2 + c_{55}l_3^2)},\\
v_g &= v_{\text{env}} = v_e = \sqrt{v_p^2 + \left(\frac{dv_p}{d\theta}\right)^2}\\
&= \frac{1}{\rho v_p}\sqrt{c_{66}^2 l_1^2 + c_{55}^2 l_3^2},\\
\tan\psi &= \frac{v_p l_1 + (dv_p/d\theta) l_3}{v_p l_3 - (dv_p/d\theta) l_1} = (c_{66}/c_{55})\tan\theta,
\end{aligned} \tag{C-1}$$

where v_g is the group velocity. It can easily be shown that

$$\begin{aligned}
v_p &= \sqrt{\frac{1}{\rho}\cdot\frac{\tan^2\psi/c_{66} + 1/c_{55}}{1/c_{55}^2 + \tan^2\psi/c_{66}^2}},\\
v_g &= \sqrt{\frac{1}{\rho}\cdot\frac{1+\tan^2\psi}{\tan^2\psi/c_{66} + 1/c_{55}}},\\
\frac{dv_g}{d\psi} &= \frac{1}{\rho v_g}\cdot\frac{\sin\psi}{\cos^3\psi}\cdot\frac{c_{55}^{-1} - c_{66}^{-1}}{(\tan^2\psi/c_{66} + 1/c_{55})^2}.
\end{aligned} \tag{C-2}$$

Lossy anisotropic medium: Homogeneous waves

In the lossy case, we have (Carcione, 2015)

$$\begin{aligned}
v_p &= \mathrm{Re}^{-1}\left(\frac{1}{v_c}\right),\\
v_c &= \sqrt{\frac{p_{66}l_1^2 + p_{55}l_3^2}{\rho}},\\
\mathbf{v}_e &= \frac{v_p}{\mathrm{Re}(v_c)}\left[\mathrm{Re}\left(\frac{p_{66}}{\rho v_c}\right) l_1\hat{\mathbf{e}}_1 + \mathrm{Re}\left(\frac{p_{55}}{\rho v_c}\right) l_3\hat{\mathbf{e}}_3\right],\\
\tan\psi_e &= \frac{\mathrm{Re}(p_{66}/v_c)}{\mathrm{Re}(p_{55}/v_c)}\tan\theta,\\
\mathbf{v}_{\text{env}} &= v_p^2\left[\mathrm{Re}\left(\frac{p_{66}}{\rho v_c^3}\right) l_1\hat{\mathbf{e}}_1 + \mathrm{Re}\left(\frac{p_{55}}{\rho v_c^3}\right) l_3\hat{\mathbf{e}}_3\right],\\
\tan\psi_{\text{env}} &= \frac{\mathrm{Re}(p_{66}/v_c^3)}{\mathrm{Re}(p_{55}/v_c^3)}\tan\theta.
\end{aligned} \tag{C-3}$$

Lossy anisotropic medium: Inhomogeneous waves

In this case, we have (Carcione and Cavallini, 1995a)

$$\begin{aligned}
v_p &= \sqrt{a_R - q^2 b_R + 2qc_I},\\
s_R &= 1/v_p,\\
\bar{\alpha} &= q s_R,\\
\rho a &= p_{66}l_1^2 + p_{55}l_3^2,\\
\rho b &= p_{66}m_1^2 + p_{55}m_3^2,\\
\rho c &= p_{66}l_1 m_1 + p_{55}l_3 m_3,\\
q &= a_I\left(c_R + \sqrt{c_R^2 + a_I b_I}\right)^{-1},\\
\mathbf{v}_e &= \frac{2\mathrm{Re}(p_{66}s_x\hat{e}_1 + p_{55}s_z\hat{e}_3)}{\rho[1 + v_p^{-2}\mathrm{Re}(a + q^2 b)]},\\
\tan\psi_e &= \frac{\mathrm{Re}(p_{66}s_x)}{\mathrm{Re}(p_{55}s_z)},\\
s_x &= s_R l_1 - \mathrm{i}\bar{\alpha} m_1,\\
z &= s_R l_3 - \mathrm{i}\bar{\alpha} m_3,\\
\mathbf{v}_{\text{env}} &= (\sin\psi_{\text{env}}, \cos\psi_{\text{env}})^\top v_{\text{env}},\\
\tan(\psi_{\text{env}} - \theta) &= \frac{1}{v_p}\frac{dv_p}{d\theta},
\end{aligned} \tag{C-4}$$

where v_{env} is calculated numerically. In this case, Snell'S law requires that $s_R \sin\theta$ and $\bar{\alpha}\sin(\theta+\gamma)$ be continuous across the interface.

APPENDIX D

qP- AND qS-WAVE EQUATIONS IN LOSSY ANISOTROPIC MEDIA

Homogeneous waves

For homogeneous waves, the propagation and attenuation directions coincide. Again, we omit, for simplicity, the subindices 1 and 2 corresponding to each medium. In this case, the horizontal complex slowness is

$$s_x = \frac{l_1}{v_c}, \tag{D-1}$$

where v_c is the complex velocity defined by

$$\rho v_c^2 = \frac{1}{2}(p_{55} + p_{11} l_1^2 + p_{33} l_3^2 \pm C), \tag{D-2}$$

with

$$C = \sqrt{[(p_{33} - p_{55}) l_3^2 - (p_{11} - p_{55}) l_1^2]^2 + 4 l_1^2 l_3^2 (p_{13} + p_{55})^2}. \tag{D-3}$$

The + and − signs correspond to the qP- and qS-waves, respectively.

To find the energy velocity and direction in lossy media, we have to compute the Umov-Poynting vector and energy densities (Carcione, 2015). The phase velocity is

$$v_p = \mathrm{Re}^{-1}\left(\frac{1}{v_c}\right) \tag{D-4}$$

and the magnitude of the energy velocity is

$$v_e = v_p / \cos(\psi_e - \theta), \tag{D-5}$$

where

$$\tan \psi_e = \frac{\mathrm{Re}(a^* \mathcal{X} + b^* \mathcal{W})}{\mathrm{Re}(a^* \mathcal{W} + b^* \mathcal{Z})}, \tag{D-6}$$

with

$$\begin{aligned} a &= \sqrt{C \pm B}, \\ b &= \pm \mathrm{pv}\sqrt{C \mp B}, \\ B &= p_{11} l_1^2 - p_{33} l_3^2 + p_{55}(l_3^2 - l_1^2), \end{aligned} \tag{D-7}$$

where the upper and lower signs correspond to the qP- and qS-waves, respectively, and

$$\begin{aligned} \mathcal{X} &= a p_{11} s_x + b p_{13} s_z, \\ \mathcal{W} &= p_{55}(b s_x + a s_z), \\ \mathcal{Z} &= a p_{13} s_x + b p_{33} s_z. \end{aligned} \tag{D-8}$$

Moreover,

$$\begin{aligned} \mathbf{v}_{\mathrm{env}} &= (\sin \psi_{\mathrm{env}}, \cos \psi_{\mathrm{env}})^\top v_{\mathrm{env}}, \\ \tan(\psi_{\mathrm{env}} - \theta) &= \frac{1}{v_p}\frac{d v_p}{d\theta}. \end{aligned} \tag{D-9}$$

Inhomogeneous waves

The procedure to obtain the equations in this case follows that of the SH-wave, but the problem has to be solved numerically. We consider the dispersion relation

$$\begin{aligned} D \equiv {} & (p_{11} s_x^2 + p_{55} s_z^2 - \rho)(p_{33} s_z^2 + p_{55} s_x^2 - \rho) \\ & - (p_{13} + p_{55})^2 s_x^2 s_z^2 = 0. \end{aligned} \tag{D-10}$$

We then use (equation A-2) and solve for s_R and $\bar{\alpha}$ from

$$\mathrm{Re}[D(s_R, \bar{\alpha})] = 0, \qquad \mathrm{Im}[D(s_R, \bar{\alpha})] = 0. \tag{D-11}$$

Equations D-11 is solved using the Newton-Raphson method for a nonlinear system of equations.

The energy-velocity vector and ray angle are obtained as

$$\mathbf{v}_e = \frac{\mathrm{Re}(\beta^* X + \xi^* W)\hat{\mathbf{e}}_1 + \mathrm{Re}(\beta^* W + \xi^* Z)\hat{\mathbf{e}}_3}{\mathrm{Re}(s_x)\mathrm{Re}(\beta^* X + \xi^* W) + \mathrm{Re}(s_z)\mathrm{Re}(\beta^* W + \xi^* Z)} \tag{D-12}$$

and

$$\tan \psi_e = \frac{\mathrm{Re}(\beta^* X + \xi^* W)}{\mathrm{Re}(\beta^* W + \xi^* Z)}, \tag{D-13}$$

where

$$\begin{aligned} W &= p_{55}(\xi s_x + \beta s_z), \\ X &= \beta p_{11} s_x + \xi p_{13} s_z, \\ Z &= \beta p_{13} s_x + \xi p_{33} s_z, \end{aligned} \tag{D-14}$$

$$\beta = \mathrm{pv}\sqrt{\frac{p_{55} s_x^2 + p_{33} s_z^2 - \rho}{p_{11} s_x^2 + p_{33} s_z^2 + p_{55}(s_x^2 + s_z^2) - 2\rho}}, \tag{D-15}$$

and

$$\xi = \pm \mathrm{pv}\sqrt{\frac{p_{11} s_x^2 + p_{55} s_z^2 - \rho}{p_{11} s_x^2 + p_{33} s_z^2 + p_{55}(s_x^2 + s_z^2) - 2\rho}}. \tag{D-16}$$

In general, the + and − signs correspond to the qP- and qS-waves, respectively.

Other equations that should hold are

$$v_e = v_p / \cos(\psi_e - \theta), \quad \text{and} \quad v_p = 1/s_R. \tag{D-17}$$

Moreover, we use equation D-9 to obtain the envelope velocity.

APPENDIX E

FERMAT'S PRINCIPLE ACCORDING TO HEARN AND KREBES

Hearn and Krebes (1990) assume a superposition of plane waves of the form

$$\exp[i\omega(t-\tau)], \qquad \tau = \mathbf{s}\cdot\mathbf{x}, \tag{E-1}$$

where $\mathbf{s} = s_x\hat{\mathbf{e}}_1 + s_z\hat{\mathbf{e}}_3$ is the slowness vector, $\mathbf{x}$ is the position vector, and τ is a complex traveltime associated to the phase of the wave. Let us consider SH-waves, whose dispersion relation is

$$D = p_{55}s_z^2 + p_{66}s_x^2 - \rho = 0. \tag{E-2}$$

This complex traveltime corresponding to the ray tracing shown in Figure 1a is

$$\begin{aligned}\tau &= s_x(x_r - x_s) + (z_s - z)s_z + (z - z_r)s_z' \\ &= s_x(x_r - x_s) + (z_s - z)\sqrt{\frac{\rho - p_{66}s_x^2}{p_{55}}} \\ &\quad + (z - z_r)\sqrt{\frac{\rho' - p_{66}'s_x^2}{p_{55}'}},\end{aligned} \tag{E-3}$$

where Snell law $s_x' = s_x$ has been assumed.

Hearn and Krebes (1990), as Richards (1984), find an s_x compatible with Fermat's principle by minimizing the complex traveltime with respect to the unknown s_x, i.e., from $d\tau/ds_x = 0$. This involves the evaluation of the Sommerfeld wavefield integral by the method of steepest descent, giving the asymptotic ray approximation. The minimization yields

$$\begin{aligned}x_r = x_s &+ (z_s - z)\sqrt{\frac{p_{66}}{p_{55}}}\frac{s_x}{\sqrt{\frac{\rho}{p_{66}} - s_x^2}} \\ &+ (z - z_r)\sqrt{\frac{p_{66}'}{p_{55}'}}\frac{s_x}{\sqrt{\frac{\rho}{p_{66}'} - s_x^2}}.\end{aligned} \tag{E-4}$$

Solving for s_x, they obtain θ_1 and γ_1 that satisfy Snell's law and Fermat's principle based on the phase time $\mathrm{Re}(\tau)$.

For qP-qS waves, the equations are more complicated and it is required a numerical solution even in the case of a homogeneous medium (e.g., from source to interface). The left part of equation E-3, imposing the condition $d\tau/ds_x = 0$, yields

$$x_r - x_s = -(z_s - z)\frac{ds_z}{ds_x} - (z - z_r)\frac{ds_z'}{ds_x}. \tag{E-5}$$

Using equation D-10, equation E-5 becomes

$$x_r - x_s = (z_s - z)\frac{\partial D/\partial s_x}{\partial D/\partial s_z} + (z - z_r)\frac{\partial D'/\partial s_x}{\partial D'/\partial s_z'}. \tag{E-6}$$

The simultaneous solution of equations D-10 and E-6 gives the slowness components for the stationary ray.

In the case of a homogeneous medium, e.g., from the source to the interface, we have

$$\begin{aligned}\frac{x - x_s}{z_s - z} &= \tan\psi = \frac{\partial D/\partial s_x}{\partial D/\partial s_z} \\ &= \left(\frac{s_x}{s_z}\right)\frac{\Gamma_{11}p_{55} + \Gamma_{33}p_{11} - \Gamma_{13}^2/s_x^2}{\Gamma_{11}p_{33} + \Gamma_{33}p_{55} - \Gamma_{13}^2/s_z^2},\end{aligned} \tag{E-7}$$

where

$$\begin{aligned}\Gamma_{11} &= p_{11}s_x^2 + p_{55}s_z^2 - \rho, \\ \Gamma_{33} &= p_{33}s_z^2 + p_{55}s_x^2 - \rho, \\ \Gamma_{13} &= (p_{13} + p_{55})s_x s_z.\end{aligned} \tag{E-8}$$

Equations D-10 and E-7 can be solved numerically to obtain s_x. Hearn and Krebes (1990) do not define an energy or ray velocity as Carcione (2015) and Vavryčuk (2007) do. However, if $t = \mathrm{Re}(\tau)$ is the traveltime, it is clear that their ray velocity is $r/\mathrm{Re}(\tau)$, where r is the distance from source to receiver along the ray. In a homogeneous medium, where $\mathbf{r} = (x - x_s, z_s - z)^\top$ (from source to interface in Figure 1), it can easily be shown that this velocity is

$$v_{\mathrm{HK}} = (\sin\psi s_{xR} + \cos\psi s_{zR})^{-1}, \tag{E-9}$$

where $\tan\psi = (x - x_s)/(z_s - z)$. In particular, equation E-9 is also valid for s_x, a solution of equation E-4 (see equation F-6).

On the other hand, it is clear that using equation A-2, we have

$$\begin{aligned}t = \mathrm{Re}(\tau) &= s_{xR}(x - x_s) + (z_s - z)s_{zR} \\ &= s_R[l_1(x - x_s) + l_3(z_s - z)] \\ &= \frac{1}{v_p}[l_1(x - x_s) + l_3(z_s - z)].\end{aligned} \tag{E-10}$$

At unit time, this equation is equivalent to equation A-4 and it can be deduced from (equation A-5) that

$$v_{\mathrm{HK}} = v_{\mathrm{env}}. \tag{E-11}$$

Further verification comes from the fact that equation E-9 can be rewritten as

$$\begin{aligned}v_{\mathrm{HK}}(\sin\psi l_1 + \cos\psi l_3) &= v_{\mathrm{HK}}(\sin\psi\,\sin\theta + \cos\psi\,\cos\theta) \\ &= v_{\mathrm{HK}}\cos(\psi - \theta) = v_p,\end{aligned} \tag{E-12}$$

which is equation A-8.

APPENDIX F

VAVRYČUK'S APPROACH TO RAY TRACING

Vavryčuk (2007, 2010, section 4) applies the steepest-descent method to derive the asymptotic Green function and finds that the ray direction is defined by an homogeneous energy-velocity vector, where in general, the slowness vector is inhomogeneous. For simplicity, let us illustrate the method with SH-waves. Using our notation, the energy-velocity vector is (Vavryčuk [2007], equation 11)

$$\mathbf{v} = \frac{1}{\rho}(p_{66}s_x, p_{55}s_z)^\top. \tag{F-1}$$

The magnitude of this vector is complex and is defined by $v = \sqrt{\mathbf{v}\cdot\mathbf{v}^\top}$, giving

$$\mathbf{v} = \frac{1}{\rho}\sqrt{p_{66}^2 s_x^2 + p_{55}^2 s_z^2} = \frac{1}{\rho}\sqrt{\rho p_{55} + p_{66}(p_{66} - p_{55})s_x^2}, \tag{F-2}$$

where we have used the dispersion equation E-2.

The ray velocity is then (Vavryčuk [2007], equation B-2)

$$v_{\mathrm{ray}} = \frac{v_R^2 + v_I^2}{v_R}. \tag{F-3}$$

To show that the Hearn and Krebes and Vavryčuk approaches are equivalent, let us consider equation E-4 giving the stationary slowness and the propagation from the source to the interface. Then,

$$\frac{x - x_s}{z_s - z} = \frac{(v_{\mathrm{env}})_x}{(v_{\mathrm{env}})_z} = \tan\psi = \sqrt{\frac{p_{66}}{p_{55}}}\frac{s_x}{\sqrt{\frac{\rho}{p_{66}} - s_x^2}} = \frac{v_x}{v_z}, \tag{F-4}$$

where we have used equations A-5, F-1, and

$$s_z = \sqrt{\frac{\rho - p_{66}s_x^2}{p_{55}}}. \tag{F-5}$$

Therefore, both approaches are equivalent if v_x/v_z is a real quantity, which occurs for

$$s_x^2 = \left(\frac{p_{55}}{p_{66}}\right)\frac{\rho\tan^2\psi}{p_{55}\tan^2\psi + p_{66}}. \tag{F-6}$$

Now, if v_x/v_z is real, we can set $v_z = bv_x$, where b is a real number. Then,

$$\hat{\mathbf{N}} \equiv \frac{\mathbf{v}}{v} = \frac{(1, b)^\top}{\sqrt{1 + b^2}} \tag{F-7}$$

is real, and the energy velocity vector is homogeneous as shown by Vavryčuk (2007, 2010). He also shows that if the stationary slowness vector is replaced into the energy-velocity vector $\mathbf{v}_e$ (equation C-4), its magnitude is v_{ray}. Then, we have that

$$v_{\mathrm{HK}} = v_{\mathrm{env}} = v_{\mathrm{ray}} = v_e \tag{F-8}$$

for the stationary slowness. For arbitrary values of θ and γ (nonstationary slowness), it is $v_{\mathrm{HK}} = v_{\mathrm{env}} \neq v_{\mathrm{ray}} \neq v_e$. However, the energy velocity vector is homogeneous only in homogeneous media. It is not homogeneous if the source and receiver lie in different media.

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