Dynamics of fluctuations in the Gaussian model with conserved dynamics

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Abstract.

We study the fluctuations of the Gaussian model, with conservation of the order parameter, evolving in contact with a thermal bath quenched from inverse temperature β_i to a final one β_f. At every time there exists a critical value $s_c(t)$ of the variance s of the order parameter per degree of freedom such that the fluctuations with $s > s_c(t)$ are characterized by a macroscopic contribution of the zero wavevector mode, similarly to what occurs in an ordinary condensation transition. We show that the probability of fluctuations with $s < \inf_t[s_c(t)]$, for which condensation never occurs, rapidly converges towards a stationary behavior. By contrast, the process of populating the zero wavevector mode of the variance, which takes place for $s > \inf_t[s_c(t)]$, induces a slow non-equilibrium dynamics resembling that of systems quenched across a phase transition.

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1. Introduction

The theory of large deviations deals with the probability of observing atypical and largely improbable events in statistical systems. Fundamental results in this branch of probability theory bear important consequences in several fields of science [1–3] and are successfully applied to various practical situations [4,5]. Under general conditions, the probability $P(S)$ to observe a certain value S of a collective variable obeys a large deviation principle [3], i.e., $P(S) \sim e^{-VI(s)}$, where V is a measure of the number of degrees of freedom contributing to S, assumed to be large, $s = S/V$ is the intensive variable associated with S, and $I(s)$ the so-called *rate function* which is non-negative and it generically vanishes at the average and most probable value of s. The above holds in the large V limit. For example, S could be thought of as being an extensive macroscopic variable, such as the number of particles in an open system of large volume V at a certain chemical potential and therefore s is the particle density in that volume.

If the conditions for the applicability of the central limit theorem are verified, small fluctuations of S of order $\sqrt{V}$ are Gaussianly distributed around its average $\langle S\rangle = \langle s\rangle V$ and hence $I(s)$ is quadratic around the most probable outcome $\langle s\rangle$ of s. On the other hand, the large deviation principle describes the rare fluctuations of S of order V which are exponentially suppressed as V increases and which can display a wealth of different and interesting behaviors. Notably, $P(S)$ can exhibit singular points [6–22] at which some derivatives are discontinuous. This fact is usually interpreted as a phase transition occurring at the level of fluctuating configurations. Namely, if s_c is one of these singular points, the configurations of the system corresponding to $s < s_c$ or to $s > s_c$ are qualitatively different. This is exactly what occurs when an ordinary phase transition is present in a statistical system. The difference is that in the latter case the typical and statistical properties of the system change qualitatively when a control parameter (the role of which is played here by s) crosses a critical value (the analogous of s_c), whereas here there is no need to change any external parameter, because rare fluctuation spontaneously occurring with $s < s_c$ or $s > s_c$ naturally correspond to radically different system properties.

In spite of the fact that large deviation theory has been widely used for studying the stationary properties of both equilibrium and non-equilibrium stochastic processes [3], the topic of the dynamics of large fluctuations is largely unexplored. The most general problem consists in understanding how an atypical state which realizes a rare fluctuation can be reached by the system starting from a certain, specified condition. A concrete example is that of two identical containers of total volume V each containing a number N of molecules of a gas, in the same thermodynamic conditions. If at some time $t = 0$ they are connected by a pipe which allows the exchange of particles between the two containers, the objective is to find the probability to observe an improbable number $S \gg \langle S\rangle = N$, e.g., $S \simeq 1.5N$, of particles in one of the two containers after an elapsed time t.

In a previous paper [23], this issue was addressed in a solvable model where $S = \sum_{k=1}^{V} s_k$ is the sum of a large number V of independent and identically distributed variables s_k, which evolve in time according to a certain stochastic dynamics. Depending on the actual distribution of the s_k, the probability $P(S)$ can exhibit a singular point S_c. Starting from a typical state with $S = \langle S\rangle$, the probability $P(S,t)$ of finding any value S was determined. It was observed that the evolution of $P(S,t)$ is radically different if a critical point S_c for the variable S is present or not.

In its absence, $P(S,t)$ evolves quite smoothly and, in a relatively short time, rare fluctuations with $S - \langle S\rangle \sim \mathcal{O}(V)$ are developed such that the probability to observe them quickly attain its stationary value. If, instead, a critical point S_c is present, the evolution occurs as described above only on one side of the value S_c (in that concrete example for $S < S_c$), whereas on the other side, the evolution of $P(S,t)$ is slow and characterized by a never-ending algebraic relaxation which strongly resembles the one observed in thermodynamic systems brought across a phase transition [24–27]. This fact reinforces the interpretation of a singular point in $P(S,t)$ as a sort of a phase transition.

In this paper we study the dynamics of fluctuations in a prototypical model of statistical mechanics, i.e., the Gaussian model. In this system, the probability distribution of the variance S of the order parameter displays a critical point S_c both in and out of equilibrium [6, 23, 28, 28–34], where the model experiences a condensation transition at the level of fluctuations, a phenomenon which has been termed *condensation of fluctuations*. Accordingly, this is a natural candidate to study how the presence of such a singularity affects the dynamical properties of large deviations, similarly to what was done in Ref. [23].

We study here the evolution of $P(S,t)$ when the system is initially prepared in an equilibrium configuration at a certain temperature β_i^{-1} and is subject at time $t=0$ to a quench, i.e., it is let subsequently to evolve with a dynamics corresponding to a different temperature β_f^{-1}. Differently from the case considered in Ref. [23], with this protocol, large deviations associated with condensed states of the system are present at any time. However, due to the abrupt change in the thermal conditions, a non-condensed configuration associated with a certain value of S can happen to cross S_c during its non-equilibrium evolution. Interpreting S_c as a critical point, also such crossing represents the occurrence of a phase transition and therefore we expect it to result into a complex and slow kinetics, as discussed above. By solving exactly the evolution equations of the model we show that this is actually what happens. Specifically, the evolution is trivial and quasi-adiabatical for fluctuations associated with a value S which does not cross S_c during the temporal evolution, while it is much richer and slow if it does.

This paper is organized as follows: In Sec. 2 we introduce the Gaussian model and its dynamics, considering in particular the quench protocol. In Sec. 3 we determine the probability $P(S,t)$ and, in Sec. 4 we discuss the condensation transition. Section 5 presents the main results concerning the evolution of $P(S,t)$, which is discussed in detail. Finally, we draw our conclusions and highlight some additional questions and open points in Sec. 6.

2. The model

We consider the Gaussian model [35, 36], describing a scalar and real field $\varphi(\vec{x})$ (an order parameter in the language of phase transitions) the equilibrium properties of which are governed by a Hamiltonian in d dimensions

$$\mathcal{H}[\varphi] = \frac{1}{2}\int_V d\vec{x}\left[(\nabla\varphi)^2 + r\varphi^2(\vec{x})\right] = \sum_{\vec{k}} \mathcal{H}_{\vec{k}}, \tag{1}$$

where $r \geq 0$ is the parameter which controls the extent $\xi = r^{-1/2}$ of the spatial correlations of the field in equilibrium, corresponding to criticality at $r=0$. On the r.h.s.,

$$\mathcal{H}_{\vec{k}} = \frac{1}{2V}\omega_k \varphi_{\vec{k}}\varphi_{-\vec{k}}, \tag{2}$$

describes $\mathcal{H}$ in terms of the Fourier components $\varphi_{\vec{k}}$ of the order parameter, where $\omega_k = k^2 + r$, and V is the volume occupied by the system. Because of the fineteness of the volume the modes are quantized, therefore the sum on the r.h.s. of Eq. (1), and although the choice of boundary conditions is inconsequential in the present problem, we assume them to be periodic. We further impose an ultraviolet cut-off Λ accounting for a microscopic length scale, due for example to a lattice spacing, such that the allowed modes are all those with wavevectors of magnitude smaller than the cut-off.

Notice that reality of the order parameter field implies that only one half of its Fourier components are independent, i.e., that $\varphi_{-\vec{k}} = \varphi^*_{\vec{k}}$. We take this into account by letting $\vec{k}$ in $\sum_{\vec{k}}$ take values only on one half of the $\vec{k}$ space and multiplying by a factor 2. Accordingly, $\mathcal{H}_{\vec{k}}$ in Eq. (2) is replaced by

$$\mathcal{H}_{\vec{k}} = \frac{1}{V}\chi_k\omega_k \varphi_{\vec{k}}\varphi_{-\vec{k}}, \tag{3}$$

where we introduced the function χ_k, such that $\chi_0 = 1/2$ and $\chi_k = 1$ otherwise, which avoids counting twice the zero mode.

The dynamics of the model with local conservation of the order parameter is given by the following overdamped Langevin evolution [35, 37]

$$\frac{\partial\varphi(\vec{x},t)}{\partial t} = -\nabla^2\left[\nabla^2 - r\right]\varphi(\vec{x},t) + \eta(\vec{x},t), \tag{4}$$

where $\eta(\vec{x},t)$ is assumed to be an uncorrelated Gaussian noise of thermal origin, at temperature β^{-1}, with zero average and

$$\langle\eta(\vec{x},t)\eta(\vec{x}',t')\rangle = -2\beta^{-1}\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t'). \tag{5}$$

With this choice of dynamics, the stationary probability distribution function of the fluctuating field is generically an equilibrium one and is given by $P_{\rm eq}[\varphi] \propto {\rm e}^{-\beta\mathcal{H}[\varphi]}$. In Fourier space one has

$$\frac{\partial\varphi_{\vec{k}}(t)}{\partial t} = -\widetilde{\omega}_k\varphi_{\vec{k}}(t) + \eta_{\vec{k}}(t), \tag{6}$$

with $\widetilde{\omega}_k = k^2(k^2+r)$ and where the noise correlator is

$$\langle\eta_{\vec{k}}(t)\eta_{\vec{k}'}(t')\rangle = \frac{V}{\chi_k}\beta^{-1}k^2\delta_{\vec{k},-\vec{k}'}\delta(t-t'). \tag{7}$$

In the following we will consider the dynamics induced by a sudden temperature quench form an initial inverse temperature value $\beta_i = (k_B T_i)^{-1}$ (k_B being the Boltzmann constant), to a final one $\beta_f > \beta_i$, operated at $t = 0$. We emphasize here that the choice of a final temperature larger than the initial one, i.e., $\beta_f < \beta_i$, leads to a different phenomenology compared to that discussed further below, which deserves a separate discussion beyond the scope of the present work.

The explicit solution of the evolution equation (6) of each mode reads, for $t \geq 0$,

$$\varphi_{\vec{k}}(t) = \varphi_{\vec{k}}(0)e^{-\tilde{\omega}_k t} + \int_0^t dt'\; e^{-\tilde{\omega}_k(t-t')}\zeta_{\vec{k}}(t'). \tag{8}$$

The correlation of the fields is therefore

$$\langle\varphi_{\vec{k}}(t)\varphi_{-\vec{k}}(t)\rangle = \langle\varphi_{\vec{k}}(0)\varphi_{-\vec{k}}(0)\rangle_0 e^{-2\tilde{\omega}_k t} + \frac{\beta_f^{-1}V}{2\chi_k\omega_k}(1-e^{-2\tilde{\omega}_k t}), \tag{9}$$

where $\langle ...\rangle_0$ stands for the average over initial conditions. This implies that the instantaneous expectation value of the Hamiltonian is

$$2\langle \mathcal{H}_{\vec{k}}\rangle = \beta_k^{-1}(t) = \left(\beta_i^{-1} - \beta_f^{-1}\right) e^{-2\widetilde{\omega}_k t} + \beta_f^{-1}, \tag{10}$$

where $\beta_k(t)$ has the heuristic meaning of a mode-dependent instantaneous non-equilibrium inverse temperature which interpolates between the initial k-independent value $\beta_k(0) = \beta_i$ and the final one $\beta_{k\neq 0}(t \to \infty) = \beta_f$, determined by the equipartition theorem in equilibrium conditions. During the non-equilibrium evolution, the equipartition theorem does not hold and, in fact, the expectation value $\langle \mathcal{H}_{\vec{k}}\rangle$ is not related to any temperature and it is mode-dependent. Note that the effective temperature $\beta_k^{-1}(t)$ is not necessarily a positive quantity and that its value at $k = 0$ is fixed by the initial condition due to the conservation law of the order parameter (and therefore we will write β_0 instead of $\beta_0(t)$ in the following).

3. Fluctuations of the variance

We will study the fluctuations of the order parameter variance

$$\mathcal{S}[\varphi] = \int_V d\vec{x}\, \varphi^2(\vec{x}, t) = \frac{2}{V} \sum_{\vec{k}} \chi_k \varphi_{\vec{k}}(t) \varphi_{-\vec{k}}(t). \tag{11}$$

The probability distribution of the value S of this quantity reads

$$P(S, t) = \int_\Gamma D\varphi\, P([\varphi], t)\, \delta(S - \mathcal{S}[\varphi]), \tag{12}$$

where Γ is the space of configurations of the field φ, $P([\varphi], t)$ is the probability of one of such configurations at time t, and δ is the Dirac delta function.

In equilibrium conditions at inverse temperature β one has $P([\varphi], t) = P_{eq}([\varphi]) = Z^{-1} e^{-\beta \mathcal{H}[\varphi]}$, where Z is the normalization constant. It is easy to show that, considering equilibrium states at different temperatures, one has (see Appendix A)

$$P_{eq}(S) = f\left(\frac{S}{\langle S\rangle}\right), \tag{13}$$

where $\langle S\rangle = \int_0^\infty dS\, S\, P(S) = \beta^{-1} \sum_{\vec{k}} \omega_k^{-1}$, is the average value of S. The scaling property (13) means that the only effect on $P_{eq}(S)$ of considering different temperatures is to set a different scale $\langle S\rangle$ of S. Accordingly, by measuring S in units of $\langle S\rangle$ one recovers the same universal behavior described by the function f reported in Eq. (13).

Because the problem is diagonalized in Fourier components, the phase-space measure $P([\varphi], t) = \Pi_{\vec{k}} P_{\vec{k}}(\varphi_{\vec{k}}, t)$ is factorized at all times. On the basis of the explicit solution for the field at a certain time given in Eq. (8), it follows that the distribution of the single $\varphi_{\vec{k}}$ are Gaussian and therefore they are completely characterized by their (vanishing) average and variance, the latter being essentially encoded in $\mathcal{H}_{\vec{k}}$, the expectation value of which is reported in Eq. (10). Thus

$$P_{\vec{k}}(\varphi_{\vec{k}}, t) = Z_{\vec{k}}^{-1}(t) e^{-\beta_k(t) \mathcal{H}_{\vec{k}}(\varphi_{\vec{k}})}, \tag{14}$$

where $Z_{\vec{k}}^{-1}(t) = \left[\frac{\chi_k \beta_k(t) \omega_k}{\pi V}\right]^{\frac{1}{2}}$.

Expressing the δ function constraint in Eq. (12) via the representation $\delta(y) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} dz\, e^{-zy}$ one arrives at

$$P(S,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} dz\, e^{-V[zs+\lambda(z,t)]}, \tag{15}$$

where $s = S/V$ is the intensive variable associated with S, and

$$\lambda(z,t) = -\frac{1}{V}\ln\int D\varphi P([\varphi],t)e^{z\mathcal{S}[\varphi]} = -\frac{1}{V}\sum_{\vec{k}}\ln\frac{1}{\sqrt{1-\frac{2z}{\beta_k(t)\omega_k}}} \tag{16}$$

is the scaled cumulant generating function. In Eq. (15), a is any real number such that $\lambda(z,t)$ is analytic for $\mathrm{Re}\, z > a$. Using Gärtner-Ellis theorem [3], for a large volume $V \to \infty$ one arrives at the large deviation form

$$P(S,t) \sim e^{-VI(s,t)}, \tag{17}$$

where the rate function $I(s,t)$ is given by

$$I(s,t) = z^*(s,t)s + \lambda(z^*(s,t),t), \tag{18}$$

where $z^*(s,t)$ is determined by the extremization condition

$$\left.\frac{\partial\lambda(z,t)}{\partial z}\right|_{z=z^*(s,t)} + s = 0. \tag{19}$$

4. Condensation

In the large volume limit, if the sums over the wavevector $\vec{k}$ can be transformed into an integral according to $\frac{1}{V}\sum_{\vec{k}}\ldots \to \int\frac{d\vec{k}}{(2\pi)^d}\ldots$, where d is the number of spatial dimensions, the extremal condition (19) reads

$$s = \Omega_d\int_0^\Lambda \frac{dk}{(2\pi)^d}\,\frac{k^{d-1}}{\beta_k(t)\omega_k - 2z^*}, \tag{20}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d-dimensional solid angle, $\beta_k(t)$ is ginven in Eq. (10), and $\Gamma(\ldots)$ the Euler function. This equation has to be solved in order to determine $z^* = z^*(s,t)$. Since s is positive by definition, z must be smaller than $\beta_0(t)\omega_0/2$, because, given Eq. (10), β_0 is the smallest among the $\beta_k(t)$ upon varying k. Notice that in the scenario with $\beta_f < \beta_i$, β_0 is no longer the smallest and this is the main reason why in this case the resulting dynamics is markedly different, as pointed out before. The integral on the r.h.s. of Eq. (20) diverges in the limit $z \to \beta_0\omega_0/2$ if $d \leq 2$, while it is finite for $d > 2$. In the latter case the solution of Eq. (20) exists only for values of s smaller than $s_c(t)$ defined by the condition

$$s_c(t) = \Omega_d\int_0^\Lambda \frac{dk}{(2\pi)^d}\,\frac{k^{d-1}}{\beta_k(t)\omega_k - \beta_0\omega_0}. \tag{21}$$

For $s > s_c(t)$, the solution requires a careful mathematical treatment [29]. Alternatively, the solution can also be found within an approach motivated and inspired by what is known for the Bose-Einsten condensation: One singles out the mode $k = 0$ from the momentum sum, transforming the rest into an integral as before, thus arriving at

$$s = \frac{1}{V}\,s_0(s,t) + \Omega_d\int_0^\Lambda \frac{dk}{(2\pi)^d}\,\frac{k^{d-1}}{\beta_k(t)\omega_k - 2z^*}, \tag{22}$$

instead of Eq. (20), with

$$s_0(s,t) = \frac{1}{\beta_0\omega_0 - 2z^*(s,t)}. \tag{23}$$

For $s < s_c(t)$,b one has $z^*(s,t) \le \beta_0\omega_0/2$ and hence the first term is negligible for large V. For $s \ge s_c(t)$, instead, one has $z^* \equiv \beta_0\omega_0/2$ and, for $s > s_c(t)$ this term becomes macroscopically large and takes the value $s - s_c(t)$. As a consequence, the large deviation form (17) holds with

$$I(s,t) = \begin{cases} z^*(s,t)s + \lambda(z^*(s,t),t) & \text{for} \quad s \le s_c(t), \\ \beta_0\omega_0(s-s_c)/2 + I(s_c,t) & \text{for} \quad s > s_c(t), \end{cases} \tag{24}$$

instead of Eq. (19). Because $I(s,t)$ is linear for $s \ge s_c(t)$ while it is not for $s \le s_c(t)$, the left and right derivatives with respect to s at $s = s_c(t)$ differ at a certain order, lager than the first one [29]. Notice also that $\lim_{s\to 0} I(s,t) = \infty$‡, hence $P(S=0,t) = 0$, because $S = 0$ can be realized by the sole configuration $\varphi \equiv 0$.

5. Dynamics of fluctuations

In the following we will study the dynamics of the fluctuations after the quench of the inverse temperature β of the stochastic noise from β_i to β_f. The evolution of $I(s,t)$, in the sample case $d = 3$, is shown in Fig. 1 for three different values of times, i.e., $t = 0$, corresponding to the initial state, $t = 0.5$ and $t = \infty$, the latter corresponding to the eventual stationary state. According to the large deviation form (17), the average value $\langle s(t)\rangle$ corresponds to the minimum, which is also the zero, of $I(s,t)$ and its expression derives from Eq. (20) taking in account the fact that, for the average, z^* in (18)vanish at all times

$$\langle s(t)\rangle = \frac{\Omega_d}{(2\pi)^d}\int_0^\Lambda dk\, \frac{k^{d-1}}{\beta_k(t)\omega_k}. \tag{25}$$

Since the fluctuations of the order parameter are due to thermal fluctuations, their typical value $\langle s(t)\rangle$ moves from the initial to the final equilibrium values $\langle s\rangle^{(eq,\beta_i)}$, $\langle s\rangle^{(eq,\beta_f)}$, obtained taking respectively $t = 0$ and $t \to \infty$ in (25), decreasing in time being $\beta_f > \beta_i$. Using the model equation it can be shown (see Appendix B) that the evolution of the average variance for sufficiently long times is

$$\langle s(t)\rangle = \langle s\rangle^{(eq,\beta_f)} + A\, t^{-d/2}, \tag{26}$$

with

$$A = \frac{\Omega_d(\beta_i^{-1} - \beta_f^{-1})\Gamma(d/2)}{r(2\pi)^d(2r)^{d/2+1}}, \tag{27}$$

where Γ is the Gamma function. Because of the Gaussian nature of the problem the critical point $s_c(t)$, above which condensation occurs, must also decrease proportionally to what $\langle s\rangle$ does. Solving the model equations (see Appendix C) one finds that during the non-equilibrium evolution $s_c(t)$ decreases monotonically and, at long times, one has

$$s_c(t) = s_c^{(eq,\beta_f)} + a\, t^{-d/2}, \tag{28}$$

‡ Indeed it can be easily checked from Eq. (20) that $z^*(s,t) \to -\infty$ with $z^*(s,t)s \to$ const. and that $\lim_{s\to 0}\lambda(z^*(s,t),t) = \infty$, after Eq. (16).

with

$$a = \frac{\Omega_d \beta_f \Gamma(d/2)\zeta(d/2)}{(2\pi)^d (2r)^{d/2+1} \beta_i (\beta_f - \beta_i)}, \tag{29}$$

where ζ is the Riemann zeta function.

During the process, the slope $\beta_0 \omega_0/2$ of the linear branch of $I(s,t)$ corresponding to condensation (see Eq. (24)) is fixed because, as already observed, β_0 is time-independent. This means that, in the condensed region for $s > s_c(\infty)$, the rate function $I(s,\infty)$ at $t = \infty$ cannot be superimposed on the initial one $I(s,0)$ using the equilibrium relation (13). The reason of this apparent incongruence is that the two equilibrium states are different not only because of β, as implicit in Eq. (13), but also because of the reduction of the set of possible final states that can be reached, starting from an assigned inital one, by the conserved dynamics.

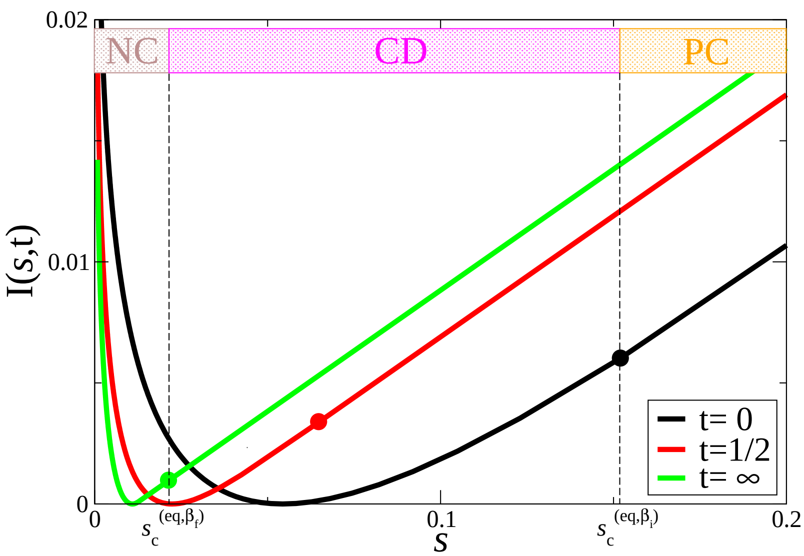

Figure 1. Rate function $I(s,t)$ as a function of s for three different values of the time $t = 0$, 0.5, and ∞ elapsed from the quench with $\beta_i = 1/5$ and $\beta_f = 1$, in the case $d = 3$ (the same qualitative features are observed for other values of $d > 2$), with $r = 1$, while the value of the ultraviolet cut-off Λ is set to 1. The critical value $s_c(t)$ of the variable s is marked by a thick dot. The three regions NC, CD, and PC, discussed in the main text, are highlighted at the top of the figure.

Given this phenomenology, it is clear that the evolution of $I(s,t)$ displays different features depending on whether condensation occurs or not. Indeed we argue below that the dynamical process accompanying condensation, namely the building up of a macroscopic $s_0(t)$ out of a microscopic initial value $s_0(t = 0)$, is much slower and collective than the easier rearrangement of fluctuations occurring at values of s for which this does not occur. On the basis of these considerations we can divide the range of values of s into three different within which fluctuations have markedly different character, as also indicated in Fig. 1.

5.1. Non-condensed (NC) region

This region corresponds to $s < s_c^{(eq,\beta_f)} = s_c(t = \infty)$ and is characterized by the fact that condensation never occurs during the dynamics and all the fluctuating modes s_k contribute to the final value $s = \sum_{\vec{k}} s_k$ of the variance with "microscopic" contributions of order $1/V$. Accordingly, during the dynamics, one simply observes the redistribution of their contributions in order for the fluctuations to pass smoothly from the initial to the final equilibrium behaviors. Give that such a redistribution involves only modes which provide microscopic contributions – contrary to what happens when condensation occurs – we expect the dynamics within this NC region to be fast.

We rationalise this hypoteses as follows: in a system at equilibrium, the scaling in Eq. (13) holds true. Clearly, the same does not hold *a priori* out of equilibrium and, indeed, there is no way to show it as one does in the case of equilibrium discussed in Appendix A. However, if the process of rearrangement occurs *quasi adiabatically*, we would expect the only effect of the quench on $I(s,t)$ to be the shift of $\langle s(t)\rangle$, according to Eq. (26), without affecting the form of $f(y)$ reported in Eq. (13). In this case, plotting $I(s,t)$ for a fixed time t as a function of $s/\langle s(t)\rangle$, one should observe superposition of the curves at different times on the mastercurve $f(y)$, formally corresponding to the case $t = \infty$. This scenario is tested in Fig. 2, where one clearly sees that in the NC region (namely to the left of the thick dot in the figure) curves corresponding to different times superimpose almost perfectly at all times, implying an adiabatic evolution.

Clearly, the scaling encoded in Eq. (13) and observed in the NC region is not expected to be exact, as in equilibrium, but it anyhow turns out to be an excellent approximation. In particular, Eq. (13) does not hold out of equilibrium because now in Eq. (A.1) there is an explicit time dependence in $\langle \psi_{\vec{k}}(t)\psi_{-\vec{k}}(t)\rangle$, where $\psi_{\vec{k}} = \langle s(t)\rangle^{-\frac{1}{2}}\,\varphi_{\vec{k}}$ is the rescaled field (see Appendix A). The observed approximate scaling behavior might be possibly due to the fact that the domain of integration in Eq. (A.1), given by the part of Γ where the argument of the δ-function vanishes, for $s < s_c$ constrains the integration variables $\psi_{\vec{k}}$ in regions much smaller than their variances $\omega_k^{-1}\langle \psi_{\vec{k}}(t)\psi_{\vec{k}}(t)\rangle$, thereby making the time-dependence induced by the dynamics largely irrelevant. Clearly this is not possible in the presence of condensation since the variance of the mode with $k = 0$ grows macroscopic.

5.2. Condensation-developing (CD) region

Any fixed value of s within the CD region $s_c^{(eq,\beta_f)} < s < s_c^{(eq,\beta_i)}$ is crossed by $s_c(t)$ at a certain time $t^*(s)$. Note that $s_c^{(eq,\beta_f)} < s_c^{(eq,\beta_i)}$ holds due to Eq. (21). This implies that for $t < t^*(s)$, the contribution to the average variance of the zero mode $s_0(s,t)$ is a finite quantity in the thermodynamic limit, with $s_0 \sim \mathcal{O}(V^0)$. Instead, for $t > t^*(s)$, $s_0(s,t)$ diverges in the same limit, with $s_0 \sim \mathcal{O}(V)$. This is pictorially sketched in Fig. 3, where in the lower panel the time behavior of $s_0(s,t)$ for various values $(s_1, s_2, s_3, s_4, s_5)$ of s within the CD region is shown; in the upper panel of the

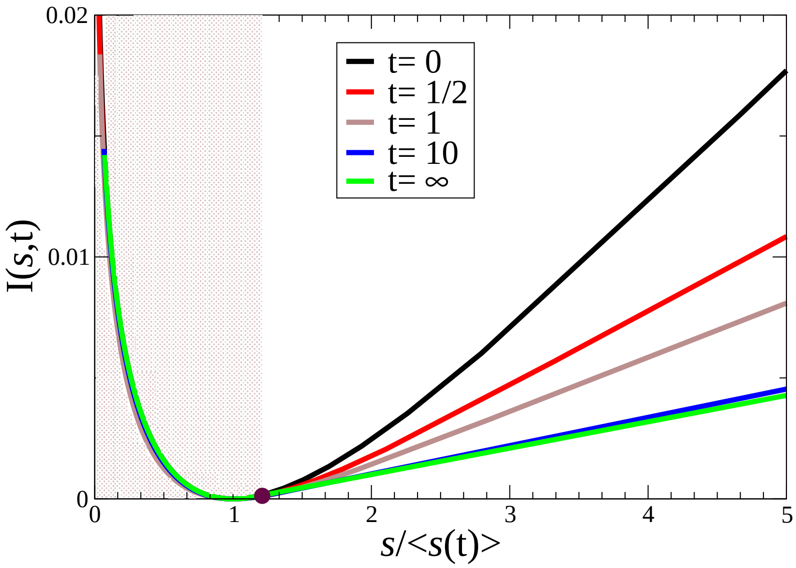

Figure 2. Rate function $I(s,t)$ as a function of the rescaled variable $s/\langle s(t)\rangle$ for various fixed values of the time t after a quench from $\beta_i = 1/5$ to $\beta_f = 1$, with $r = 1$, $\Lambda = 1$, in the case $d = 3$. The critical value $s_c^{(eq,\beta_f)}/\langle s\rangle^{(eq,\beta_f)}$ is marked by a thick dot and the NC region is highlighted by a brown background.

same figure the position of these values is shown in relationship with the rate functions at $t = 0$ and at $t = \infty$, the critical values of which (indicated by the dots) define the boundaries of the CD region. For times $t \lesssim t^*(s)$ the divergence of $s_0(s,t)$ occurs as (see Appendix D)

$$\lim_{V\to\infty} s_0(s,t) \simeq \begin{cases} [t^*(s) - t]^{-1} & \text{for} \quad s > s_c^{(eq,\beta_f)}, \\ t^{d/2} & \text{for} \quad s = s_c^{(eq,\beta_f)}, \end{cases} \tag{30}$$

i.e., $s_0(s,t)$ with $s > s_c^{(eq,\beta_f)}$ diverges linearly while $s_0(s_c^{(eq,\beta_f)}, t)$ algebraically.

Figure 2 shows that the relaxation of the rate function in the CD region is much slower than that in the NC region. Indeed, while for the times reported in the figure fluctuations are adiabatically in equilibrium in the NC region (corresponding to the values of s on the left of the thick dot), in the CD and in the further region denoted as PC (discussed below) where condensation is present from the beginning (on the right of the dot) a significant change is observed and convergence occurs only at much longer times ($t \gtrsim 10$ on the scale of the present figure).

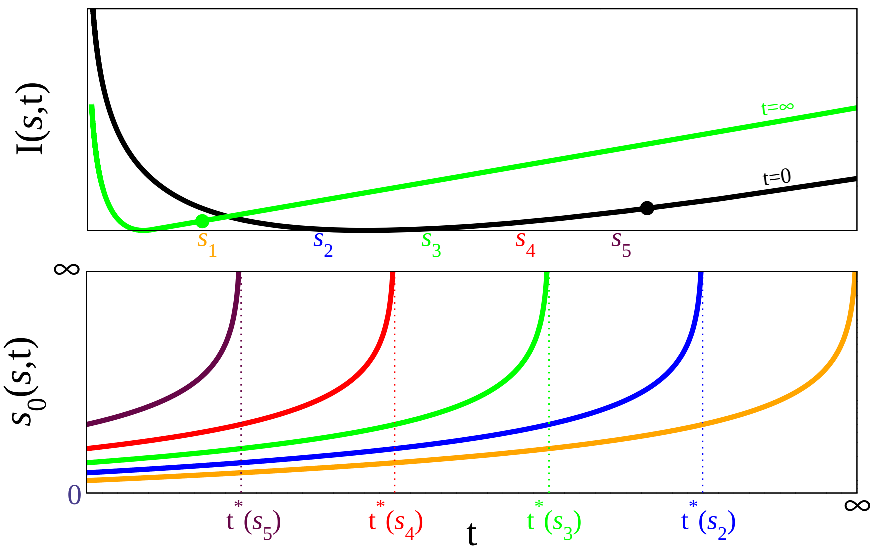

Figure 3. Upper panel: Rate function $I(s,t)$ as a function of s the initial equilibrium state , i.e., immediately before the quench (black line, $t=0$) and in the final one ($t=\infty$). In both cases, the corresponding critical value of s are indicated by dots. Lower panel: time dependence of $s_0(s,t)$, for various values $(s_1, s_2, s_3, s_4, s_5)$ of s which, for comparison, are located in the upper panel with respect to the rate functions at $t=0$ and $t=\infty$. Both panels refer to the case $d=3$, $r=1$ and $\Lambda=1$ for a quench from $\beta_i=1/5$ to $\beta_f=1$.

5.3. Permanent-condensation (PC) region

For $s > s_c^{(eq,\beta_i)}$, $s_0(s,t)/V$ increases monotonically in time from $s - s_c^{(eq,\beta_i)}$ to $s - s_c^{(eq,\beta_f)}$. Also in this case $s_0(s,t)$ changes by an infinite amount in the thermodynamic limit $V\to\infty$. This is similar to what happens when the value of s is within the CD region described above, apart from the fact that in the latter case $s_0(s,0)$ is finite. Accordingly, we observe also in this case that fluctuations do not relax adiabatically. Notice also that, no matter how large t is, for sufficiently large values of s the rate function $I(s,t)$ differs significantly from its asymptotic form. A similar behavior was observed in Ref. [23].

6. Conclusions

In this paper we have analysed some aspects of the dynamics of fluctuations of the variance s per degree of freedom, of the order parameter in the Gaussian model with a conserved stochastic dynamics, in which large deviations may display the phenomenon of condensation. After a quench of the temperature of the thermal bath the model is in contact with, we have shown that the non-equilibrium behavior of fluctuations is radically different depending on whether the selected value of s is affected or not by the condensation as time goes by. In particular, fluctations which do not condense converge almost adiabatically to a stationary, equilibrium-like form. Those affected by the condensation, instead, display a slow and complex evolution determined by the slow contribution $s_0(s,t)$ of the $k=0$ wavevector.

The emergence of these two qualitatively different behaviors, which was already observed in another solvable model [23] of statistical mechanics, has a nice interpretation in the framework of what is known for ordinary phase transitions. It must be recalled, in fact, that the expression (11) of the probability we consider is formally equivalent [3, 28] to the partition function of a Gaussian model on a reduced phase space where the order-parameter variance is fixed to take the value S. This correspondence is usually referred to as *duality*. A well-known model with such a constraint is the spherical model of Berlin and Kac [38]. This model has a ferromagnetic to paramagnetic phase transition located at $s_c(\beta)$. Crossing a critical point in magnetic models induces a slow, never-ending (in the thermodynamic limit) coarsening phenomenon characterized by an algebraic growth of a quantity that sets the scale of spatial fluctuations. Indeed, the zero wavevector mode of the structure factor diverges because of the formation of the Bragg peak, $\lim_{t\to\infty}\langle\varphi(\vec{k},t)\varphi(-\vec{k},t)\rangle \sim \delta(\vec{k})$. In the problem considered in this work, the values of s within the CD region are crossed, at a certain time, by $s_c(t)$ and therefore they are expected to share some of the properties of the slow kinetics observed in quenched ferromagnets. In fact, we have shown that this is actually the case, and the quantity $s_0(s_c^{(eq,\beta_f)},t)$ diverges algebraically. Clearly, relaxation in the NC region is much faster, corresponding — according to the analogy drawn above — to quenching a ferromagnetic system without crossing the critical point.

In the present work we focussed on a particular kind of quench, in which the temperature β^{-1} of the thermal bath responsible for the stochastic noise is changed abruptly. It must be noticed that letting the initial value $\beta_i \to 0$ implies that $s_c^{(eq,\beta_i)}$ grows to infinity and, accordingly, no condensation occurs in the initial state. This case, which is recovered as a special limit of the solution presented in this work, is more closely related to what was done in Ref. [23] where condensation is initially absent as well.

While we studied here the case of a quench of the temperature of the thermal bath, one might consider different kind of quenches, e.g., those in which other parameters are varied, such as r or, equivalently, the coefficient of the square gradient term in Eq. (1) (which we fixed here to be one for simplicity). Similarly, other observables beyond the order-parameter variance could be considered. Apart from quantitative specific differences, we expect to observe in all these cases phenomena similar to those described here, with markedly different behavior of fluctuations depending on whether they cross or not a critical point. Analogously, they are expected in the Gaussian model with purely relaxational dynamics, i.e., without conservation of the order parameter, with the notable difference that, in this case, the relaxation occurs exponentially fast in time, in contrast to the algebraic one observed in the present model (see for instance Eqs. (30), (26) and (28)).

The model considered here, and the related cases discussed above, as well as the model considered in Ref. [23] are characterized by independently fluctuating modes. However, there are examples of probability distributions which display a behaviour similiar to the one discussed in this work also in more complex systems in which these modes interact, for instance in intrinsically non-equilibrium states of models of active matter [30, 39]. The dynamics of fluctuations in these cases is largely unexplored and represents an interesting topic for further investigations.

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Appendix A.

In this Appendix we prove the scaling property in Eq. (13) for the equilibrium distribution function $P_{eq}(S)$ of the variable S in Eq. (11). Starting from Eq. (12) we change variable as $(\langle S(t)\rangle/V)^{1/2}$ one has

$$\begin{aligned} P(S,t) = \quad & Z^{-1}(t) \int_\Gamma D\psi \, \exp\left\{-\frac{1}{2V}\sum_{\vec k}\omega_k \frac{\psi_{\vec k}\psi_{-\vec k}}{\langle\psi_{\vec k}(t)\psi_{-\vec k}(t)\rangle}\right\}\times \\ & \delta\left(\frac{1}{V}\sum_{\vec k}\psi_{\vec k}\psi_{-\vec k}-\frac{S}{\langle S(t)\rangle}\right). \end{aligned} \tag{A.1}$$

In equilibrium all the time dependences drop out, and $\langle\psi_{\vec k}\psi_{-\vec k}\rangle$ is independent of the temperature (and of $\vec k$), due to the equipartition theorem. Hence one has Eq. (13).

Appendix B.

In this Appendix we derive the expression for the evolution at long times of the average of s. In the large-volume limit we have, from Eq. (25)

$$\begin{aligned} \langle s(t)\rangle \quad & = \frac{\Omega_d}{(2\pi)^d}\int_0^\Lambda dk\, \frac{k^{d-1}}{\beta_k(t)\omega_k} \\ & = \frac{\Omega_d}{(2\pi)^d}\int_0^\Lambda dk \frac{k^{d-1}((\beta_i^{-1}-\beta_f^{-1})e^{-2k^2(k^2+r)t}+\beta_f^{-1})}{k^2+r}. \end{aligned} \tag{B.1}$$

The final equilibrium value, obtained for $t\to\infty$ in the previous expression, reads

$$\langle s\rangle^{(eq,\beta_f)} = \frac{\Omega_d}{(2\pi)^d}\int_0^\Lambda dk\,\frac{k^{d-1}}{\beta_f(k^2+r)}. \tag{B.2}$$

The difference $\langle s(t)\rangle - \langle s\rangle^{(eq,\beta_f)}$ is therefore given by

$$\langle s(t)\rangle - \langle s\rangle^{(eq,\beta_f)} = \frac{\Omega_d(\beta_i^{-1}-\beta_f^{-1})}{(2\pi)^d}\int_0^\Lambda dk\,\frac{k^{d-1}e^{-2k^2(k^2+r)t}}{k^2+r}. \tag{B.3}$$

Changing variable $x=t^{\frac{1}{2}}k$ leads to

$$\langle s(t)\rangle - \langle s\rangle^{(eq,\beta_f)} = \frac{\Omega_d(\beta_i^{-1}-\beta_f^{-1})t^{-d/2}}{(2\pi)^d}\int_0^{\Lambda\sqrt{t}} dx\,\frac{x^{-d/2}e^{-2x^2(x^2/t+r)}}{x^2/t+r}. \tag{B.4}$$

For large t, due to the fact that only small x contribute, the integral can be written as

$$\langle s(t)\rangle - \langle s\rangle^{(eq,\beta_f)} \simeq \frac{\Omega_d(\beta_i^{-1}-\beta_f^{-1})t^{-d/2}}{r(2\pi)^d}\int_0^\infty dx\, x^{-d/2}e^{-2x^2 r}. \tag{B.5}$$

Accordingly, one recovers Eq. (26), with

$$\begin{aligned} A &= \tfrac{\Omega_d(\beta_i^{-1}-\beta_f^{-1})}{r(2\pi)^d}\int_0^\infty dx\, x^{-d/2}e^{-2x^2r} \\ &= \tfrac{\Omega_d(\beta_i^{-1}-\beta_f^{-1})\Gamma(d/2)}{r(2\pi)^d(2r)^{d/2+1}}, \end{aligned} \tag{B.6}$$

where Γ is the Gamma function. In order to assess the accuracy of this approximation for large t we evaluated numerically $\langle s(t)\rangle$ in the case $d = 3$, finding almost perfect correspondence.

Appendix C.

In this Appendix we determine the evolution at large times of the critical point s_c at long times, proceeding as in Appendix B. Here we have, again in the large-volume limit, starting from Eq. (21)

$$s_c(t) - s_c^{(eq,\beta_f)} = \frac{\Omega_d}{(2\pi)^d}\int_0^\Lambda dk\, k^{d-1}\Big[\frac{1}{(k^2+r)\beta_k(t) - r\beta_i} - \frac{1}{(k^2+r)\beta_f - r\beta_i}\Big], \tag{C.1}$$

where $\beta_k(t)$ is defined in (10). Changing variables $x = t^{\frac{1}{2}}k$ one has

$$s_c(t) - s_c^{(eq,\beta_f)} = \frac{\Omega_d t^{-d/2}}{(2\pi)^d}\int_0^{\Lambda\sqrt{t}} dx\, x^{d-1}\Big[\frac{1}{\frac{x^2/t+r}{(\beta_i^{-1}-\beta_f^{-1})e^{-2x^2(x^2/t+r)}+\beta_f^{-1}} - r\beta_i} - \frac{1}{(x^2/t+r)\beta_f - r\beta_i}\Big]. \tag{C.2}$$

For large t we end up with

$$s_c(t) - s_c^{(eq,\beta_f)} \simeq \frac{\Omega_d t^{-d/2}}{(2\pi)^d}\int_0^\infty dx\, x^{d-1}\Big[\frac{1}{\frac{r}{(\beta_i^{-1}-\beta_f^{-1})e^{-2x^2r}+\beta_f^{-1}} - r\beta_i} - \frac{1}{r(\beta_f-\beta_i)}\Big], \tag{C.3}$$

namely Eq. (28). The value of the coefficient a introduced therein is

$$\begin{aligned} a &= \tfrac{\Omega_d}{(2\pi)^d}\int_0^\infty dx\, x^{d-1}\Big[\tfrac{1}{\frac{r}{(\beta_i^{-1}-\beta_f^{-1})e^{-2x^2r}+\beta_f^{-1}}-r\beta_i} - \tfrac{1}{r(\beta_f-\beta_i)}\Big] \\ &= \tfrac{\Omega_d\beta_f\Gamma(d/2)\zeta(d/2)}{(2\pi)^d(2r)^{d/2+1}\beta_i(\beta_f-\beta_i)}, \end{aligned} \tag{C.4}$$

where ζ is the Riemann zeta function. Numerical calculations, for the test case $d = 3$, confirm Eq. (C.3) with excellent accuracy.

Appendix D.

In this Appendix we determine the dynamics of the condensing mode $s_0(s,t)$ in the condensation-developing region. We recall that $s_0(s,t)$ is defined in Eq. (23), where $z^*(s,t)$ is defined via Eq. (19)

$$s = \Omega_d\int_0^\Lambda \frac{dk}{(2\pi)^d}\,\frac{k^{d-1}}{\beta_k(t)\omega_k - 2z^*}, \tag{D.1}$$

with ω_k defined after Eq. (2). In order to determine the behaviour of $s_0(s,t)$ as time goes by, we must first determine that of $z^*(s,t)$. Taking the time derivative of Eq. (D.1) one has

$$0 = \int_0^\Lambda dk k^{d-1}\frac{[\tilde\omega_k\omega_k\beta_k^2(t)(\beta_i^{-1}-\beta_f^{-1})e^{-2\tilde\omega_k t} - \dot z^*(s,t)]}{[\beta_k(t)\omega_k - 2z^*(s,t)]^2}, \tag{D.2}$$

where $\dot{z}^*$ stands for the time derivative of z^*, while $\beta_k(t)$ is given in Eq. (10) and $\tilde{\omega}_k$ is defined after Eq. (6). Accordingly,

$$\dot{z}^* = \frac{\int_0^\Lambda dk \frac{k^{d-1}\tilde{\omega}_k \omega_k \beta_k^2(t)(\beta_i^{-1}-\beta_f^{-1})e^{-2\tilde{\omega}_k t}}{[\beta_k(t)\omega_k - 2z^*(s,t)]^2}}{\int_0^\Lambda dk \frac{k^{d-1}}{[\beta_k(t)\omega_k - 2z^*(s,t)]^2}}. \tag{D.3}$$

In order to proceed, we distinguish between values of s inside the CD region, i.e., $s_c^{(eq,\beta_f)} < s < s_c^{(eq,\beta_i)}$ and the limiting value $s = s_c^{(eq,\beta_f)}$. In the former case, s_0 will diverge in a finite time so in the integrals in Eq. (D.3) we can consider the limit of small k, which also correspond to the portion of the domain where the variation in time is more important. Accordingly, from Eq. (D.3) one finds

$$\dot{z}^* = \frac{\int_0^\Lambda dk \, \frac{\frac{k^{d+1}r^2(\beta_i^{-1}-\beta_f^{-1})}{(\beta_i^{-1})^2}}{\left(\frac{r}{\beta_i^{-1}}-2z^*\right)^2}}{\int_0^\Lambda dk \, \frac{k^{d-1}}{\left(\frac{r}{\beta_i^{-1}}-2z^*\right)^2}} = \frac{\int_0^\Lambda dk \, \frac{k^{d+1}r^2(\beta_i^{-1}-\beta_f^{-1})}{(\beta_i^{-1})^2}}{\int_0^\Lambda dk \, k^{d-1}} = \text{const.} \tag{D.4}$$

Thus z^* is linear in t, implying

$$s_0(s,t) \simeq (t^*(s) - t)^{-1}, \qquad \forall s \in (s_c^{(eq,\beta_f)}, s_c^{(eq,\beta_i)}). \tag{D.5}$$

In the other case, $s = s_c^{(eq,\beta_f)}$ is at the border of the CD region and we can consider the limit of long times in Eq. (D.3). The integrand in the denominator, then, can be approximated by its leading behaviour for small $k \ll r^{1/2}$, i.e., as a constant

$$\beta_k(t)\omega_k - 2z^*(s_c^{(eq,\beta_f)},t) \simeq \beta_f r - 2z^*(s_c^{(eq,\beta_f)},\infty). \tag{D.6}$$

Accordingly, Eq. (D.3) in the same limit renders

$$\dot{z}^* \simeq \frac{d}{\Lambda^d}\int_0^\Lambda dk \frac{k^{d-1}k^2r^2(\beta_i^{-1}-\beta_f^{-1})e^{-2k^2rt}}{[(\beta_i^{-1}-\beta_f^{-1})e^{-2k^2rt}+\beta_f^{-1}]^2}, \tag{D.7}$$

where we used that $\tilde{\omega}_k \simeq k^2 r$ and $\omega_k \simeq r$. The change of variables $x = t^{\frac{1}{2}}k$ gives

$$\dot{z}^* \simeq \frac{t^{\frac{d+2}{2}}d}{\Lambda^d}\int_0^{\Lambda\sqrt{t}} dk \frac{x^{d+1}r^2(\beta_i^{-1}-\beta_f^{-1})e^{-2x^2r}}{[(\beta_i^{-1}-\beta_f^{-1})e^{-2x^2r}+\beta_f^{-1}]^2}. \tag{D.8}$$

In the long-time limit the integral is well approximated by the one in which the upper extreme of integration is set to infinity, and therefore $\dot{z}^* \simeq t^{-d/2+1}$. Accordingly

$$z^*(s_c^{(eq,\beta_f)},t) \simeq z^*(s_c^{(eq,\beta_f)},\infty) + Ct^{-d/2}, \tag{D.9}$$

where the asymptotic value equals $\beta_0\omega_0/2$ and C is a proportionality constant. We conclude that

$$s_0(s_c^{(eq,\beta_f)},t) \simeq t^{d/2}, \tag{D.10}$$

namely Eq. (30). We confirmed numerically the validity of this result in the specific case $d = 3$.